

流体与流动 · 流体动力学的本构方程

FLUID FLOWS · CONSTITUTIVE EQUATION OF FLUID DYNAMICS

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Constitutive Equation

OUTLINE

① DESCRIPTIONS OF FLUID FLOWS

Continuous Medium Hypothesis

Eulerian Description and Lagrangian Description

Streamline, Pathline, Streakline, Streamtube

② MATERIAL SUBSTANTIAL DERIVATIVE

③ REYNOLDS TRANSPORT THEOREM

④ HELMHOLTZ VELOCITY DECOMPOSING THEOREM

Expressions of Strain Rate and Rotation Rate Tensor

Physical Meanings of Strain Rate and Rotation Rate Tensors

Description of Fluid Flows with Vorticity

⑤ PATTERNS AND CLASSIFICATION OF FLUID FLOWS

⑥ CONSTITUTIVE EQUATION

Stress Tensor

Constitutive Equation

Viscosity Coefficients

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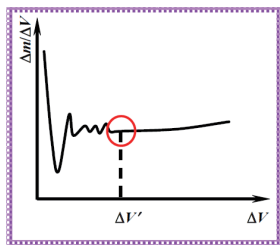
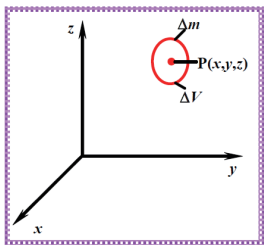
Constitutive Equation

Viscosity Coefficients

CONTINUOUS MEDIUM HYPOTHESIS

■ 连续介质假设

- 把流体看成无数多的、极小的、可以看成质点的微团连续不断紧密无隙组成的物质。



特征体积

- 流体质点微团的特征体积：“在微观上充分大、在宏观上充分小”。

CONTINUOUS MEDIUM HYPOTHESIS

■ 流体微团必须具备的两个条件

- ① 微观上足够大，流体微团必须包含足够多的流体分子，分子的个体行为对宏观特性的影响可以忽略不计。
- ② 宏观上足够小，流体微团是一个没有空间尺寸的几何点，其体积无限小。

■ 由下式判断一种流体能否按连续介质假设来研究

$$l \ll d \ll L \quad (1)$$

■ 连续介质假设的物理意义

- ① 摆脱了复杂的分子运动，实现了将微观问题转化为宏观问题来处理。
- ② 从而可利用数学分析中连续函数这个有力工具来研究流体的平衡和运动问题。

TWO WAYS TO DESCRIBE FLUID FLOWS

■ The Eulerian Description

着眼于空间点，在空间的每一点上描述流体运动随时间的变化。

$$P = P(x, y, z, t).$$

■ The Lagrangian Description

着眼于流体质点，描述每个流体质点自始至终的运动，即它的位置随时间变化。

$$P = P(x_0, y_0, z_0, t).$$

■ Connection of the Ways

这两种描述是针对同一种流体流动的两种描述方法，只是着眼点不同，在实质上是等价的。

TWO WAYS TO DESCRIBE FLUID FLOWS

■ 欧拉参考系和拉格朗日参考系中的自变量

- ① 在欧拉参考系中, x, y, z, t 是相互间无函数关系的独立变量。
- ② 在拉格朗日参考系中, x, y, z 不再是独立变量, 而是 $r_0(x_0, y_0, z_0)$ 和时间 t 的函数。

$$x - x_0 = u(t - t_0)$$

$$y - y_0 = v(t - t_0)$$

$$z - z_0 = w(t - t_0)$$

■ 欧拉法与拉格朗日法描述的转化:

$$\begin{aligned} P = P(x, y, z, t) &= P(x(x_0, y_0, z_0, t), y(x_0, y_0, z_0, t), z(x_0, y_0, z_0, t), t) \\ &= P(x_0, y_0, z_0, t). \end{aligned}$$

THE EULERIAN DESCRIPTION

② 空间中的速度分布



$$\vec{u} = \vec{u}(x, y, z, t) = \vec{u}(\vec{r}, t)$$

固定 xyz ，变化 t

某确定空间点参数
随时间的变化规律

② 空间中的压强分布、温度等的分布



$$\left\{ \begin{array}{l} p = p(x, y, z, t) = p(\vec{r}, t) \\ T = T(x, y, z, t) = T(\vec{r}, t) \end{array} \right.$$

固定 t ，变化 xyz

同一时刻不同空
间点的参数分布

场—分布着某种物理量的空间区域

流场

THE LAGRANGIAN DESCRIPTION

任意时刻流体质点的位置矢量

$$\vec{r} = \vec{r}(\vec{r}_0, t) \Rightarrow \begin{cases} x = x(x_0, y_0, z_0, t) \\ y = y(x_0, y_0, z_0, t) \\ z = z(x_0, y_0, z_0, t) \end{cases}$$

其它物理量

$$p = p(\vec{r}_0, t), \quad \rho = \rho(\vec{r}_0, t), \quad T = T(\vec{r}_0, t)$$

任意时刻流体质点的速度

$$\vec{u}(\vec{r}, t) = \frac{d\vec{r}(\vec{r}_0, t)}{dt}$$

THE LAGRANGIAN DESCRIPTION

任意时刻流体质点的加速度

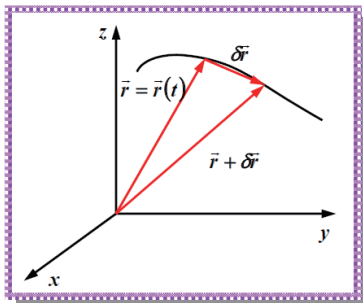


$$\vec{a} = \frac{d^2 \vec{r}(\vec{r}_0, t)}{dt^2}$$

固定 t , 变化 $\vec{r}_0$, 同一时刻不同流体质点的参数分布

固定 $\vec{r}_0$, 变化 t , 某确定流体质点随时间的运动规律

拉格朗日参考系中, 流体质点的空间位置随时间变化, 因此 x, y, z 是时间 t 的函数, x, y, z, t 不是相互独立的



FROM EULERIAN TO LAGRANGIAN

由速度求流体质点的运动轨迹

$$\leftarrow d\vec{r} = \vec{u}dt \Rightarrow \vec{r} = \vec{r}(c_1, c_2, c_3, t)$$

代入初始条件

$$\leftarrow t = t_0, \vec{r} = \vec{r}_0$$

$$\rightarrow c_1 = c_1(\vec{r}_0), c_2 = c_2(\vec{r}_0), c_3 = c_3(\vec{r}_0) \rightarrow \vec{r} = \vec{r}(\vec{r}_0, t)$$

代入欧拉参考系表达式

$$\rightarrow \vec{u} = \vec{u}[\vec{r}(\vec{r}_0, t), t] = \vec{u}(\vec{r}_0, t)$$

如已知欧拉参考系中的其他变量

$$\rightarrow \eta = \eta[\vec{r}(\vec{r}_0, t), t] = \eta(\vec{r}_0, t)$$

FROM EULERIAN TO LAGRANGIAN

例：已知速度场 $u = \frac{x}{1+t}$, $v = \frac{2y}{1+t}$, $w = \frac{3z}{1+t}$, 求矢径表达式 $\vec{r} = \vec{r}(x_0, y_0, z_0, t)$ 及加速度的拉格朗日描述

(1) 矢径表达式

$$dx = u dt = \frac{x}{1+t} dt \quad \Longrightarrow \quad \frac{dx}{x} = \frac{dt}{1+t} \quad \Longrightarrow \quad \ln x = \ln(1+t) + c$$

$$\Longrightarrow \quad x = c_1(1+t)$$

同理 $y = c_2(1+t)^2 \quad z = c_3(1+t)^3$

FROM EULERIAN TO LAGRANGIAN

由 $t = 0$ 时 $c_1 = x_0$, $c_2 = y_0$, $c_3 = z_0$ 得

$$\Rightarrow \underline{\underline{\vec{r} = x_0(1+t)\vec{i} + y_0(1+t)^2\vec{j} + z_0(1+t)^3\vec{k}}}$$

(2) 加速度的拉格朗日描述

$$a_x = \frac{\partial^2 x}{\partial t^2} = \frac{\partial^2}{\partial t^2} [x_0(1+t)] = \underline{\underline{0}}$$

$$a_y = \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2}{\partial t^2} [y_0(1+t)^2] = \underline{\underline{2y_0}}$$

$$a_z = \frac{\partial^2 z}{\partial t^2} = \frac{\partial^2}{\partial t^2} [z_0(1+t)^3] = \underline{\underline{6z_0(1+t)}}$$

FROM EULERIAN TO LAGRANGIAN

方法

已知 $\eta = \eta[\vec{r}_0, t]$ 将 $\vec{r}_0 = \vec{r}_0(\vec{r}, t)$ 代入左式



$$\eta = \eta[\vec{r}_0(\vec{r}, t), t]$$



$$\eta = \eta[\vec{r}, t]$$

例：拉格朗日变数 (x_0, y_0, z_0) 给出的流体运动规律为：
 $x = x_0 e^{-2t}$, $y = y_0 (1+t)^2$, $z = z_0 e^{2t} (1+t)^{-2}$, 求(1)以欧拉坐标表示的速度场；(2)试确定流动是否为定常；(3)求加速度

FROM EULERIAN TO LAGRANGIAN

$$x = x_0 e^{-2t}, \quad y = y_0 (1+t)^2, \quad z = z_0 e^{2t} (1+t)^{-2}$$

(1) 以欧拉坐标表示的速度场

$$u = \frac{\partial x}{\partial t} = -2x_0 e^{-2t}$$

$$v = \frac{\partial y}{\partial t} = 2y_0 (1+t) = \frac{2y_0 (1+t)^2}{(1+t)}$$

$$w = 2z_0 e^{2t} \left[(1+t)^{-2} - (1+t)^{-3} \right] = \frac{2z_0 e^{2t} (1+t)^{-2} t}{(1+t)}$$

由题中给出的运动规律，可得


$$\underline{\underline{u = 2x, \quad v = \frac{2y}{(1+t)}, \quad w = \frac{2zt}{(1+t)}}}$$

STEADY, UNSTEADY

定常场与非定常场



流场中每一点的物理量都不随时间变化，称为定常场；否则，为非定常场

@ 定常场数学描述



$$\frac{\partial \eta}{\partial t} = 0$$

或

$$\eta = \eta(x, y, z)$$

STEADY, UNSTEADY

(2) 流动是否定常

$$u = 2x, \quad v = \frac{2y}{(1+t)}, \quad w = \frac{2zt}{(1+t)}$$

速度的欧拉表达式包含变量 t , 所以是非定常流动

(3) 加速度

$$a_x = \frac{\partial u}{\partial t} = 4x_0 e^{-2t} = \underline{\underline{4x}}$$

$$a_y = \frac{\partial v}{\partial t} = 2y_0 = \underline{\underline{\frac{2y}{(1+t)^2}}}$$

$$a_z = \frac{\partial w}{\partial t} = \frac{2z_0 e^{2t}}{(1+t)^3} + \frac{4z_0 e^{2t} t}{(1+t)^3} - \frac{6z_0 e^{2t} t}{(1+t)^4} = \frac{2z_0 e^{2t} (1+2t^2)}{(1+t)^4} = \underline{\underline{\frac{2z(1+2t^2)}{(1+t)^2}}}$$

HOMOGENEOUS, HETEROGENEOUS

均匀场与非均匀场



流场中各空间点上的物理量都一样，称为均匀场；
否则，为非均匀场

④ 均匀场数学描述



$$\frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial y} = \frac{\partial \eta}{\partial z} = 0$$

或

$$\eta = \eta(t)$$

STREAMLINE, PATHLINE, STREAKLINE

■ 流线

流场中的一条**假想曲线**，在时刻 t 位于该曲线上的所有流体质点的运动方向都与此曲线相切。流线的微分方程可表示为

$$d\mathbf{r} \times \mathbf{u} = 0.$$

■ 迹线

迹线是流体质点在流场中的**运动轨迹**。迹线的微分方程可表示为

$$d\mathbf{r} = \mathbf{u}dt.$$

■ 脉线

脉线是相继通过流场同一空间点的流体质点在同一瞬时的连线。脉线也称为**染色线**。

染色是一种流场显示技术，用于反映流场结构、流动特点。

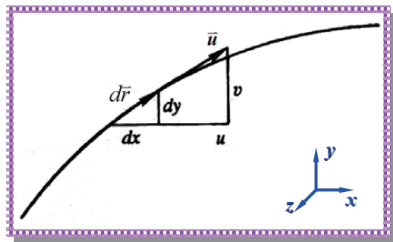
STREAMLINE, PATHLINE, STREAKLINE

🚦 流线微分方程

$$d\vec{l} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{u} = u\vec{i} + v\vec{j} + w\vec{k}$$

$$d\vec{l} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ dx & dy & dz \\ u & v & w \end{vmatrix} = 0$$



$$\frac{dx}{u(x, y, z, t)} = \frac{dy}{v(x, y, z, t)} = \frac{dz}{w(x, y, z, t)}$$

⌚ t 为常数, x, y, z, t 相互独立

STREAMLINE, PATHLINE, STREAKLINE

例：已知平面流动速度场： $u = x(1+2t)$ ， $v = y$ ， $w = 0$
 求： $t = 0$ 时刻通过 $(1, 1)$ 点的流体质点的流线

微分方程 $\Rightarrow \frac{dx}{x(1+2t)} = \frac{dy}{y}$

积分 $\Rightarrow \ln x = \ln y^{(1+2t)} + c \Rightarrow x = c_1 y^{(1+2t)}$

定解条件 $t = 0 \Rightarrow x = y = 1 \Rightarrow c_1 = 1$

$\Rightarrow x = y^{(1+2t)} \xrightarrow{t=0} \underline{\underline{x = y}}$

非定常流动，流线
随时间不同而不同

STREAMLINE, PATHLINE, STREAKLINE

迹线方程

$$\vec{u}(\vec{r}, t) = \frac{d\vec{r}}{dt}$$



$$\frac{dx}{u} = \frac{dy}{v} = \frac{dw}{w} = dt$$

积分得 $d\vec{r} = \vec{u}dt \Rightarrow \vec{r} = \vec{r}(c_1, c_2, c_3, t)$

$$t = t_0, \vec{r} = \vec{r}_0$$



$$c_1 = c_1(\vec{r}_0), c_2 = c_2(\vec{r}_0), c_3 = c_3(\vec{r}_0)$$



$$\vec{r} = \vec{r}(\vec{r}_0, t) = \vec{r}(x_0, y_0, z_0, t)$$

◎ t 是自变量, x, y, z 是流体质点的空间坐标, 因此是 t 的函数, 求解是在拉格朗日参考系中进行的

STREAMLINE, PATHLINE, STREAKLINE

例：已知平面流动速度场： $u = x(1 + 2t)$ ， $v = y$ ， $w = 0$
求： $t = 0$ 时刻通过 $(1, 1)$ 点的流体质点的迹线

微分方程 $\Rightarrow \begin{cases} \frac{dx}{dt} = x(1 + 2t) \\ \frac{dy}{dt} = y \end{cases}$

满足上述速度分布的流场中有无数个流体质点于是有无数条迹线，本题只求其中一条

积分 $\Rightarrow x = c_1 e^{t(1+t)}$ ， $y = c_2 e^t$

初始条件 $t = 0 \Rightarrow x = y = 1 \Rightarrow c_1 = c_2 = 1$

$\Rightarrow x = e^{t(1+t)}$ ， $y = e^t \Rightarrow \underline{\underline{x = y^{1+\ln y}}}$

STREAMLINE, PATHLINE, STREAKLINE

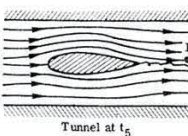
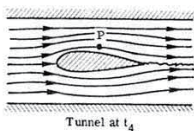
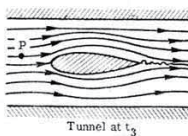
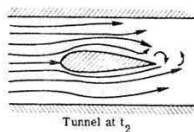
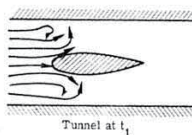
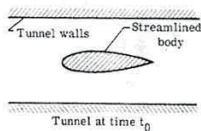
- 流线一般不可相交和不可转折，否则

STREAMLINE, PATHLINE, STREAKLINE

- 流线一般不可相交和不可转折，否则在相交点有两个速度，但在特殊的驻点处可以相交。定常流动的流线、迹线与脉线

STREAMLINE, PATHLINE, STREAKLINE

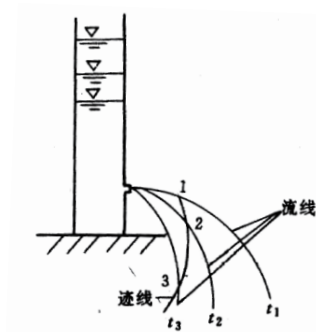
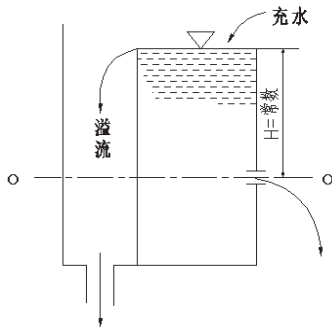
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Steady flow: Particle pathline = Streamline

STREAMLINE, PATHLINE, STREAKLINE

- 在定常流动中，流线不随时间改变其位置和形状，流线和迹线重合。



- 在非定常流动中，由于各空间点上速度随时间变化，流线的形状和位置是在不停地变化的。

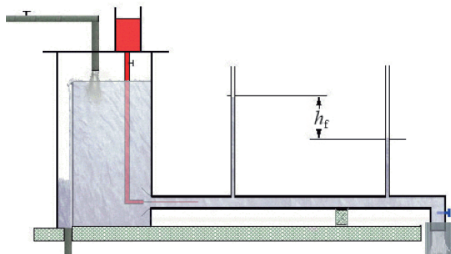
STREAMLINE, PATHLINE, STREAKLINE

脉线



相继通过流场同一空间点的
流体质点在同一瞬时的连线

◎ 流场显示技术，反映流场结构、流动特点



STREAMLINE, PATHLINE, STREAKLINE

- ④ 求 τ 时刻从 x_* 、 y_* 、 z_* 进入流场的流体质点的迹线方程，即 τ 时刻通过 x_* 、 y_* 、 z_* 的迹线

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dw}{w} = dt$$

- ④ 初始条件 $\Rightarrow$ $t = \tau$ 时, $x = x_*$, $y = y_*$, $z = z_*$

- ④ 积分上述方程 $\Rightarrow$ $\begin{cases} x = x(x_*, y_*, z_*, t, \tau) \\ y = y(x_*, y_*, z_*, t, \tau) \\ z = z(x_*, y_*, z_*, t, \tau) \end{cases}$ 或 $\vec{r} = \vec{r}(\vec{r}_*, t, \tau)$

脉线切线与速度矢量方向不一定相同

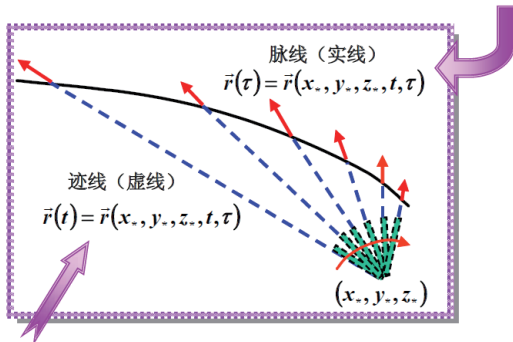
STREAMLINE, PATHLINE, STREAKLINE

$$\vec{r}(\tau) = \vec{r}(\vec{r}_*, t, \tau)$$

固定 t

τ 在 $-\infty \leq \tau \leq t$ 内取值

t 瞬时前不同时刻 τ 经由 (x_*, y_*, z_*) 点注入流场的不同流体质点在 t 时刻的空间位置



τ 时刻由点 (x_*, y_*, z_*) 注入流场的一个流体质点的迹线; 不同的 τ 表示不同的迹线

$$\vec{r}(t) = \vec{r}(\vec{r}_*, t, \tau)$$

固定 τ , t 在 $t \geq \tau$ 内取值

STREAMLINE, PATHLINE, STREAKLINE

例：已知平面流动速度场： $u = x(1+2t)$, $v = y$, $w = 0$
求： $t = 0$ 时刻通过 $(1, 1)$ 点的流体质点的脉线

微分方程 $\Rightarrow \begin{cases} \frac{dx}{dt} = x(1+2t) \\ \frac{dy}{dt} = y \end{cases}$

积分 $\Rightarrow x = c_1 e^{t(1+t)}$, $y = c_2 e^t$

初始条件 $t = \tau \Rightarrow x = y = 1$

$\Rightarrow c_1 = e^{-\tau(1+\tau)}$, $c_2 = e^{-\tau}$

脉线微分方程的通解与迹线微分方程的通解相同。

在时刻 τ ，脉线过点 $(1, 1)$ ；在时刻 0 也过点 $(1, 1)$ 。

$$-\infty \leq \tau \leq t$$

STREAMLINE, PATHLINE, STREAKLINE

例：已知平面流动速度场： $u = x(1+2t)$ ， $v = y$ ， $w = 0$
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$\Leftarrow$ 在时刻 τ ，脉线过点 $(1,1)$ ；在时刻 0 也过点 $(1,1)$ 。

$$-\infty \leq \tau \leq t$$

STREAMLINE, PATHLINE, STREAKLINE

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$\Rightarrow c_1 = e^{-\tau(1+\tau)}$ ， $c_2 = e^{-\tau}$

脉线微分方程的通解与迹线微分方程的通解相同。

在时刻 τ ，脉线过点 $(1,1)$ ；在时刻 0 也过点 $(1,1)$ 。

$-\infty \leq \tau \leq t$

STREAMLINE, PATHLINE, STREAKLINE

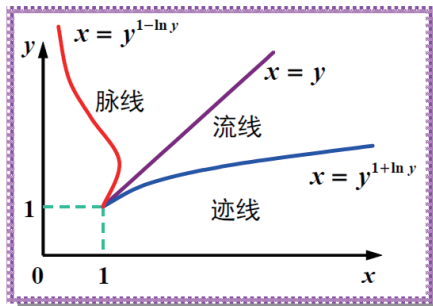
⇒ $x = e^{t(1+t)-\tau(1+\tau)}$, $y = e^{t-\tau}$

上式即通过 (1, 1) 点的脉线方程, 式中 $-\infty \leq \tau \leq t$

显然不同时刻，脉线形状不同，令 $t = 0$

⇒ $x = e^{-\tau(1+\tau)}$, $y = e^{-\tau}$

 $x = y^{1-\ln y}$

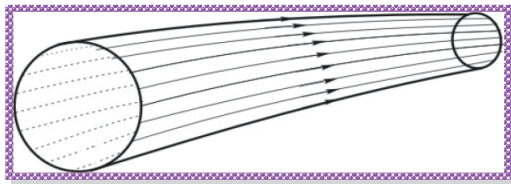


STREAMTUBE

在流场中做一封闭且不自相交的曲线 C ，在某瞬时通过该曲线上的流线构成的管状表面称为流管

④ 有限流管

④ 流管元



④ 流体不能从侧壁穿入穿出

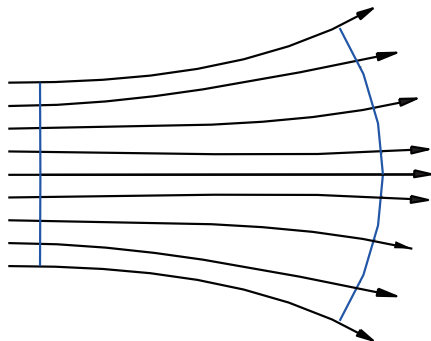
④ 定常流动时，定常流动时流管形状不变，类似于固定管道

STREAMTUBE

有效截面、流量与平均流速

- ① 有效截面：处处与流线相垂直的流束的截面 A_0
- ② 流量：单位时间内流经某一规定表面的流体量

$$\begin{aligned}
 Q_V &= \int_{A_0} \mathbf{V} \cdot d\mathbf{A} \\
 &= \int_{A_0} V_n \cdot dA
 \end{aligned}$$



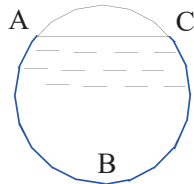
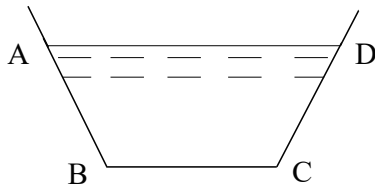
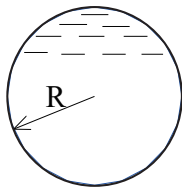
- ③ 平均流速：流经有效截面的体积流量除以有效截面积而得到的商 $\bar{V} = Q_V/A_0$ 。

STREAMTUBE

湿周、水力半径与当量直径

- ① 湿周：在有效截面上，流体同固体边界接触部分的周长 χ
- ② 水力半径：有效截面积与湿周之比称为水力半径

$$R_h = A_0 / \chi$$

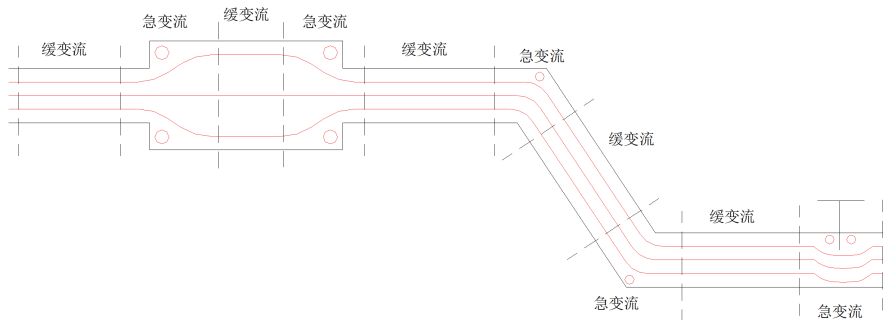


- ③ 当量直径：水力半径的4倍为当量直径， $D_e = 4A_0 / \chi$

STREAMTUBE

急变流、缓变流

① 急变流：流线间相互不平行，有夹角的流动



② 缓变流：流线平行或接近平行的流动

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SUBSTANTIAL DERIVATIVE

■ 欧拉参考系和拉格朗日参考系中的时间导数

① 欧拉参考系

$$\mathbf{u} = \mathbf{u}(x, y, z, t)$$

$$\left(\frac{\partial \mathbf{u}}{\partial t} \right)_{x,y,z} \Rightarrow \text{当地导数或局部导数：某一空间点上的流体速度随时间的变化。}$$

② 拉格朗日参考系

$$\mathbf{u} = \mathbf{u}(x_0, y_0, z_0, t)$$

$$\left(\frac{\partial \mathbf{u}}{\partial t} \right)_{x_0,y_0,z_0} \Rightarrow \text{流体质点的加速度：流体质点的速度随时间的变化率。}$$

SUBSTANTIAL DERIVATIVE

- 随体导数，又称：质点导数、物质导数。

流体质点物理量随时间的变化率。

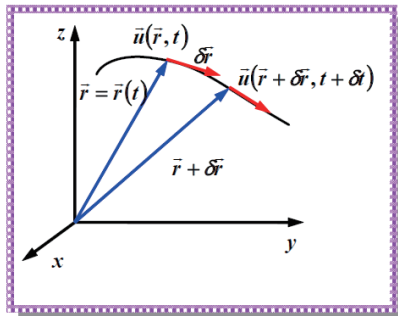
$\left(\frac{Du}{Dt}\right)_{x,y,z} \Rightarrow$ 欧拉参考系下某流体质点的速度随时间变化率。

在 t 时刻 $\Rightarrow$

$$\mathbf{u}(x, y, z, t)$$

在 $t + \delta t$ 时刻 $\Rightarrow$

$$\mathbf{u}(x + \delta x, y + \delta y, z + \delta z, t + \delta t)$$



SUBSTANTIAL DERIVATIVE

- 流体质点的速度随时间变化率

$$\begin{aligned}
 \mathbf{a} &= \lim_{\delta t \rightarrow 0} \frac{\mathbf{u}(x + \delta x, y + \delta y, z + \delta z, t + \delta t) - \mathbf{u}(x, y, z, t)}{\delta t} \\
 &= \lim_{\delta t \rightarrow 0} \frac{\mathbf{u}(\mathbf{r}, t) + \frac{\partial \mathbf{u}}{\partial x} \delta x + \frac{\partial \mathbf{u}}{\partial y} \delta y + \frac{\partial \mathbf{u}}{\partial z} \delta z + \frac{\partial \mathbf{u}}{\partial t} \delta t - \mathbf{u}(\mathbf{r}, t)}{\delta t} \\
 &= \frac{\partial \mathbf{u}}{\partial t} + u \frac{\partial \mathbf{u}}{\partial x} + v \frac{\partial \mathbf{u}}{\partial y} + w \frac{\partial \mathbf{u}}{\partial z} \\
 &= \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \\
 &= \frac{D\mathbf{u}}{Dt} \quad \Rightarrow \quad \text{欧拉参考系下流体质点的加速度}
 \end{aligned}$$

SUBSTANTIAL DERIVATIVE

① 当地加速度或局部加速度

$\frac{\partial \mathbf{u}}{\partial t} \Rightarrow$ 空间点上的流体速度随时间的变化率，
由速度场的非定常性引起。

$$\frac{\partial \mathbf{u}}{\partial t} = 0 \quad \Leftrightarrow \quad \text{速度场定常}$$

② 迁移加速度或对流加速度

$\mathbf{u} \cdot \nabla \mathbf{u} \Rightarrow$ 由流体质点在非均匀速度场中的运动引起。

$$u \frac{\partial \mathbf{u}}{\partial x} + v \frac{\partial \mathbf{u}}{\partial y} + w \frac{\partial \mathbf{u}}{\partial z} = 0 \quad \Leftrightarrow \quad \text{速度场均匀}$$

SUBSTANTIAL DERIVATIVE

■ 推广到流体质点的其他物理量 η 随时间变化率

① 当地导数或局部导数

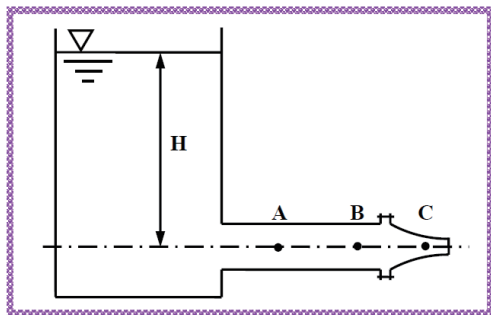
$\frac{\partial \eta}{\partial t} \Rightarrow$ 空间点上物理量 η 随时间的变化率，由物理量场的非定常性引起。

② 对流导数或位变导数

$\mathbf{u} \cdot \nabla \eta \Rightarrow$ 由流体质点在非均匀物理量场中运动引起的 η 变化率。

$$\eta \text{ 均匀分布} \Rightarrow u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} + w \frac{\partial \eta}{\partial z} = 0$$

SUBSTANTIAL DERIVATIVE



定常

$A \Rightarrow B$

匀速直线运动
无当地和对流加
速度

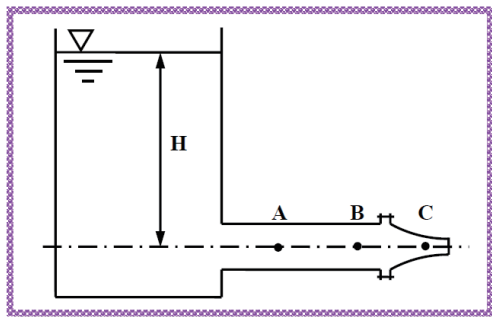
$B \Rightarrow C$

加速运动，存在
对流加速度

非定常

?

SUBSTANTIAL DERIVATIVE



定
常

$A \Rightarrow B$

匀速直线运动
无当地和对流加
速度

$B \Rightarrow C$

加速运动，存在
对流加速度

非定常

$A \Rightarrow B$

速度变化，存在
当地加速度

$B \Rightarrow C$

速度变化，存在当地
和对流加速度

SUBSTANTIAL DERIVATIVE · EXAMPLE

例：为研究城市的空气污染情况，需测量某项污染指标 s 随时间的变化率，采用了三种方法：1) 把测量探头安装在一高塔上；2) 把探头安装在一直升飞机上，直升飞机速度为 $\bar{U}$ ；3) 把探头安装在一气球上，设气球随气流运动，气流速度为 $\bar{u}$ 。试用数学公式分别表示上述三种方法的测量结果。

(1) 测量探头安装在一高塔上，测得的是在流场某一固定点上 s 的随时间的变化率，即 s 的当地导数



$$\frac{\partial s}{\partial t}$$

SUBSTANTIAL DERIVATIVE · EXAMPLE

(2) 直升飞机上探头测得的 s 变化率应等于 s 的当地变化率加上 s 的空间变化率与直升飞机速度的乘积

$$\longrightarrow \frac{\partial s}{\partial t} + U_x \frac{\partial s}{\partial x} + U_y \frac{\partial s}{\partial y} + U_z \frac{\partial s}{\partial z}$$

(3) 由于气球与空气速度相同，气球上探头测得的 s 变化率就是 s 的随体导数或物质导数

$$\longrightarrow \frac{Ds}{Dt} = \frac{\partial s}{\partial t} + u_x \frac{\partial s}{\partial x} + u_y \frac{\partial s}{\partial y} + u_z \frac{\partial s}{\partial z}$$

SUBSTANTIAL DERIVATIVE · EXAMPLE

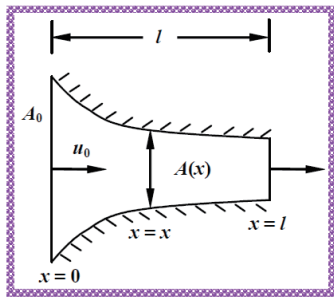
例：考虑图示收缩通道内理想不可压缩流体的一维定常流动，分别求欧拉和拉格朗日参考系内的速度和加速度表达式

解：欧拉参考系，不可压缩流体
根据连续方程

$$u_0 A_0 = u(x) A(x)$$

$$\Rightarrow u(x) = \frac{u_0 A_0}{A(x)} = u_0 \left(1 + x/l\right)$$

$$\Rightarrow a_x = \frac{Du}{Dt} = u \frac{\partial u}{\partial x} = \frac{u_0^2}{l} \left(1 + \frac{x}{l}\right)$$



$$A(x) = A_0 \left(1 + x/l\right)$$

SUBSTANTIAL DERIVATIVE · EXAMPLE

在拉格朗日参考系中， $u = u(t)$ 是 $t = 0$ 时刻从 $x = 0$ 出发的流体质点的速度表达式，由

$$u = u_0(1 + x/l) \implies \frac{dx}{dt} = u_0(1 + x/l) \implies \int_0^x \frac{dx}{1 + x/l} = \int_0^t u_0 dt$$

$$\implies l \ln(1 + x/l) = u_0 t \implies x = l \left[\exp\left(\frac{u_0 t}{l}\right) - 1 \right]$$

两边对 t 求导 $\implies u = \frac{dx}{dt} = u_0 \exp\left(\frac{u_0 t}{l}\right)$

再对 t 求导 $\implies a = \frac{du}{dt} = \frac{u_0^2}{l} \exp\left(\frac{u_0 t}{l}\right)$

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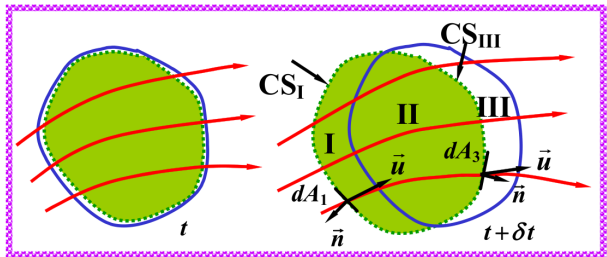
Viscosity Coefficients

REYNOLDS TRANSPORT THEOREM

■ System 系统

确定流体质点所组成某一集合的**总体**。

- ▶ 与外界无质量交换。
- ▶ 空间位置随流体质点的运动而运动。



- ▶ 系统的边界形状、空间大小随流体质点的运动而变化。
- 在拉格朗日参考系中，通常把注意力集中在流动的系统上。

THE SYSTEM · PHYSICAL THEOREMS

- 描述系统（拉格朗日方法）的常用物理定律：

① 质量守恒方程

$$m_{\text{sys}} = \text{const} \quad \text{or} \quad \frac{dm}{dt} = 0$$

② 动量方程

$$\mathbf{F} = m\mathbf{a} = m \frac{d\mathbf{u}}{dt} = \frac{d(m\mathbf{u})}{dt}$$

③ 动量矩方程

$$\mathbf{T} = \frac{d\mathbf{H}}{dt}, \quad \mathbf{H} = \sum (\mathbf{r} \times \mathbf{u}) \delta m$$

④ 能量守恒方程

$$\frac{dE}{dt} = \dot{Q} + \dot{W}$$

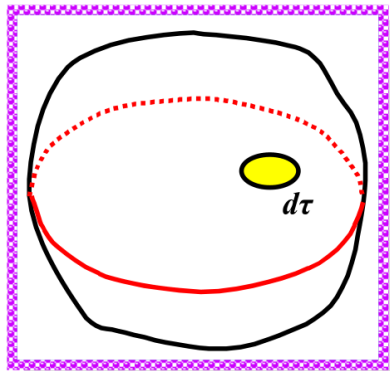
THE SYSTEM · SUBSTANTIAL DERIVATIVE

- 系统物理量体积分的随体导数

$$\frac{DN_{\text{sys}}}{Dt} = \frac{D}{Dt} \int_{\text{sys}} \phi d\tau$$

$N \Rightarrow$ 系统在所占空间包含的总物理量。

$\phi \Rightarrow$ 单位体积流体物理量的分布函数。



- 质量及其分布函数

$$N = m \quad \Rightarrow \quad \phi = \rho$$

- 动量及其分布函数

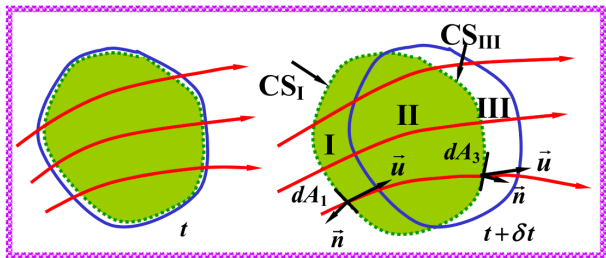
$$N = m\mathbf{u} \quad \Rightarrow \quad \phi = \rho\mathbf{u}$$

REYNOLDS TRANSPORT THEOREM

Control Volume 控制体

流场中的某一确定的空间区域。

- ▶ 与外界有质量交换。
- ▶ 空间位置相对于某参照系不变。



- ▶ 控制体边界形状、所包围的空间大小是确定的。

- 在欧拉参考系中，通常把注意力集中在通过控制体的流体上。

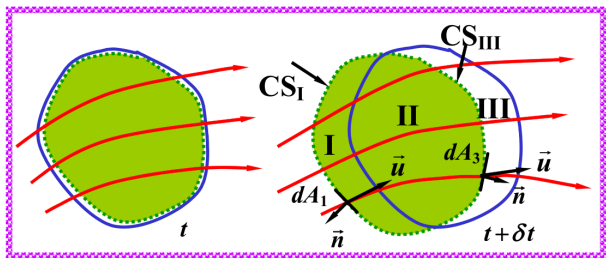
REYNOLDS TRANSPORT THEOREM

■ Reynolds Transport Theorem 雷诺输运定理

用欧拉方法描述系统物理量随时间的变化率，即采用与控制体相关的量来描述系统该物理量之体积分的随体导数。

- ▶ 初始时刻：
系统与控制体重合。

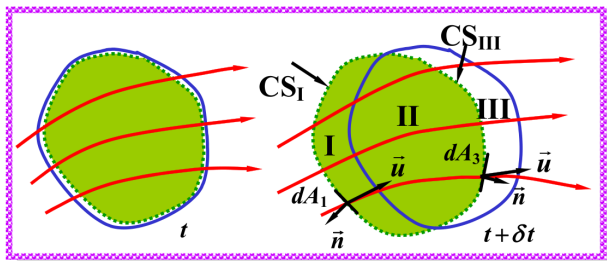
- ▶ 经过 δt 后：
控制体: I + II
系统: II + III



- 在数学上，物质导数是微分形式的表达式，雷诺输运定理是积分形式的表达式。

REYNOLDS TRANSPORT THEOREM

- ▶ 初始时刻:
系统与控制体重合。
- ▶ 经过 δt 后:
控制体: I + II
系统: II + III



$$\begin{aligned}
 \frac{DN_{\text{sys}}}{Dt} &= \lim_{\delta t \rightarrow 0} \frac{N_{\text{sys}}(t + \delta t) - N_{\text{sys}}(t)}{\delta t} \\
 &= \lim_{\delta t \rightarrow 0} \frac{N_{\text{CV}}(t + \delta t) - N_{\text{I}}(t + \delta t) + N_{\text{III}}(t + \delta t) - N_{\text{CV}}(t)}{\delta t} \\
 &= \lim_{\delta t \rightarrow 0} \frac{N_{\text{CV}}(t + \delta t) - N_{\text{CV}}(t)}{\delta t} - \lim_{\delta t \rightarrow 0} \frac{N_{\text{I}}(t + \delta t)}{\delta t} \\
 &\quad + \lim_{\delta t \rightarrow 0} \frac{N_{\text{III}}(t + \delta t)}{\delta t}
 \end{aligned}$$

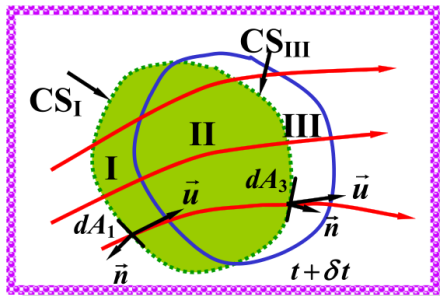
REYNOLDS TRANSPORT THEOREM

- 第1项：控制体CV所包含 N 的变化率

$$\lim_{\delta t \rightarrow 0} \frac{N_{CV}(t + \delta t) - N_{CV}(t)}{\delta t} = \frac{\partial N_{CV}}{\partial t} = \frac{\partial}{\partial t} \int_{CV} \phi d\tau$$

- 第2项：通过面 CS_I 流入CV所包含的 N 。此时，流入的流体体积为： $\delta\tau = -\mathbf{u} \cdot \mathbf{n} dA_1 \delta t$ 。

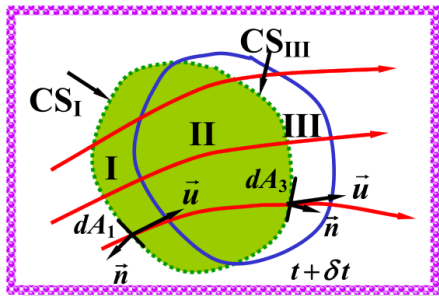
$$\begin{aligned} - \lim_{\delta t \rightarrow 0} \frac{N_I(t + \delta t)}{\delta t} &= - \lim_{\delta t \rightarrow 0} \frac{1}{\delta t} \int_{I(t+\delta t)} \phi d\tau \\ &= - \lim_{\delta t \rightarrow 0} \frac{1}{\delta t} \int_{CS_I} -\phi \mathbf{u} \cdot \mathbf{n} dA \delta t = \int_{CS_I} \phi \mathbf{u} \cdot \mathbf{n} dA \end{aligned}$$



REYNOLDS TRANSPORT THEOREM

- 第3项：通过面 CS_{III} 流出CV所包含的 N 。此时，流出的流体体积为： $\delta\tau = \mathbf{u} \cdot \mathbf{n} dA_3 \delta t$ 。

$$\begin{aligned} & \lim_{\delta t \rightarrow 0} \frac{N_{III}(t + \delta t)}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{1}{\delta t} \int_{III(t+\delta t)} \phi d\tau \end{aligned}$$



$$\Rightarrow \lim_{\delta t \rightarrow 0} \frac{N_{III}(t + \delta t)}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{1}{\delta t} \int_{CS_{III}} \phi \mathbf{u} \cdot \mathbf{n} dA \delta t = \int_{CS_{III}} \phi \mathbf{u} \cdot \mathbf{n} dA$$

$$\Rightarrow \frac{DN_{sys}}{Dt} = \frac{\partial}{\partial t} \int_{CV} \phi d\tau + \int_{CS_I} \phi \mathbf{u} \cdot \mathbf{n} dA + \int_{CS_{III}} \phi \mathbf{u} \cdot \mathbf{n} dA$$

REYNOLDS TRANSPORT THEOREM

Reynolds Transport Theorem

$$\frac{DN_{\text{sys}}}{Dt} = \frac{D}{Dt} \int_{\tau} \phi d\tau = \frac{\partial}{\partial t} \int_{\tau} \phi d\tau + \int_A \phi \mathbf{u} \cdot \mathbf{n} dA$$

- $\frac{DN_{\text{sys}}}{Dt} \Rightarrow$ 系统的物理量 N 随时间的变化率。
- $\int_{\tau} \frac{\partial \phi}{\partial t} d\tau \Rightarrow$ 控制体的物理量 N 随时间的变化率。
它反应了流场的非定常性。
- $\int_A \phi \mathbf{u} \cdot \mathbf{n} dA \Rightarrow$ 物理量 N 流出控制体的净流率。
它反应了流场的不均匀性。
系统的位置和体积均随时间发生变化。

REYNOLDS TRANSPORT THEOREM

Reynolds Transport Theorem

$$\frac{D}{Dt} \int_{\tau} \phi d\tau = \frac{\partial}{\partial t} \int_{\tau} \phi d\tau + \int_A \phi \mathbf{u} \cdot \mathbf{n} dA = \int_{\tau} \frac{\partial \phi}{\partial t} d\tau + \int_A \phi \mathbf{u} \cdot \mathbf{n} dA$$

通过高斯定理 **Gauss' Flux Theorem**

$$\int_A \phi \mathbf{u} \cdot \mathbf{n} dA = \int_{\tau} [\nabla \cdot (\phi \mathbf{u})] d\tau$$

将面积分转化为体积分

$$\Rightarrow \frac{D}{Dt} \int_{\tau} \phi d\tau = \int_{\tau} \left[\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) \right] d\tau = \int_{\tau} \left[\frac{\partial \phi}{\partial t} + \frac{\partial (\phi u_k)}{\partial x_k} \right] d\tau$$

REYNOLDS TRANSPORT THEOREM - SAMPLE

例：给定一流场的速度分布和密度分布为

$$u = x/r^3, \quad v = y/r^3, \quad w = z/r^3, \quad \rho = k(r^3 - 3t)$$

其中, $r^2 = x^2 + y^2 + z^2$, k 为非零常数。求：(1) 流场中某空间点的流体密度随时间的变化率；(2) 流体质点的密度在运动过程中随时间的变化率；(3) 初始时刻在体积 $0 \leq r \leq a$ 中的流体质量的随体导数

(1) 流场中某空间点的流体密度随时间的变化率

$$\xrightarrow{\text{green arrow}} \quad \frac{\partial \rho}{\partial t} = \underline{\underline{-3k}}$$

(2) 流体质点的密度在运动过程中随时间的变化率

$$\xrightarrow{\text{green arrow}} \quad \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z}$$

REYNOLDS TRANSPORT THEOREM - SAMPLE

$$\begin{aligned}
 \Rightarrow \frac{D\rho}{Dt} &= \frac{\partial\rho}{\partial t} + u \frac{\partial\rho}{\partial x} + v \frac{\partial\rho}{\partial y} + w \frac{\partial\rho}{\partial z} \\
 &= -3k + \frac{x}{r^3} 3kr^2 \frac{x}{r} + \frac{y}{r^3} 3kr^2 \frac{y}{r} + \frac{z}{r^3} 3kr^2 \frac{z}{r} \\
 &= -3k + 3k \left(\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} \right) = -3k + 3k = 0
 \end{aligned}$$

$$\begin{aligned}
 u &= x/r^3 \\
 v &= y/r^3 \\
 w &= z/r^3 \\
 \rho &= k(r^3 - 3t)
 \end{aligned}$$

密度的当地导数、对流导数均不为零，但密度的随体导数为零，即每个流体质点在运动过程中密度保持不变，为不可压缩流体

(3) 在体积 $0 \leq r \leq a$ 中的流体质量的随体导数

流体质量 $\Rightarrow M = \int_{\tau} \rho d\tau = \int_{\tau} k(r^3 - 3t) d\tau$

REYNOLDS TRANSPORT THEOREM - SAMPLE

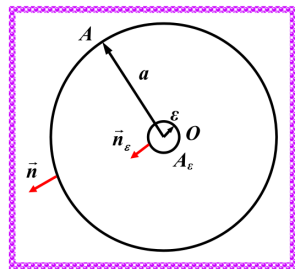
流体质量的随体导数 

$$\frac{DM}{Dt} = \frac{\partial}{\partial t} \int_{\tau} k(r^3 - 3t) d\tau + \lim_{\varepsilon \rightarrow 0} \oint_{A+A_{\varepsilon}} k(r^3 - 3t) \vec{u} \cdot \vec{n} dA$$

$$\begin{aligned} u &= x/r^3 \\ v &= y/r^3 \\ w &= z/r^3 \\ \rho &= k(r^3 - 3t) \end{aligned}$$

M 的当地导数 

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\tau} k(r^3 - 3t) d\tau &= \int_{\tau} \frac{\partial}{\partial t} [k(r^3 - 3t)] d\tau \\ &= \int_{\tau} -3k d\tau = -4k\pi a^3 \end{aligned}$$



M 的对流导数 

$$\int_{A+A_{\varepsilon}} k(r^3 - 3t) \vec{u} \cdot \vec{n} dA = \int_A k(r^3 - 3t) \vec{u} \cdot \vec{n} dA + \int_{A_{\varepsilon}} k(r^3 - 3t) \vec{u} \cdot \vec{n} dA$$

REYNOLDS TRANSPORT THEOREM - SAMPLE

由于 $\vec{u} = \frac{x}{r^3} \vec{i} + \frac{y}{r^3} \vec{j} + \frac{z}{r^3} \vec{k} = \frac{\vec{r}}{r^3}$

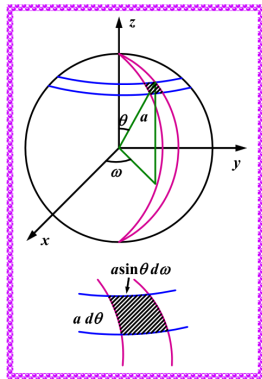
在 A 表面上 $\vec{u} \cdot \vec{n} = \frac{\vec{r}}{r^3} \cdot \frac{\vec{r}}{r} = \frac{1}{r^2} = \frac{1}{a^2}$

在 A_ε 表面上 $\vec{u} \cdot \vec{n} = -\vec{u} \cdot \vec{n}_\varepsilon = -\frac{1}{\varepsilon^2}$



$$\begin{aligned} & \int_A k(r^3 - 3t) \vec{u} \cdot \vec{n} dA \\ &= \int_0^{2\pi} d\omega \int_0^\pi k(a^3 - 3t) \frac{1}{a^2} a^2 \sin \theta d\theta \\ &= 4\pi k(a^3 - 3t) \end{aligned}$$

$$\begin{aligned} u &= x/r^3 \\ v &= y/r^3 \\ w &= z/r^3 \\ \rho &= k(r^3 - 3t) \end{aligned}$$



REYNOLDS TRANSPORT THEOREM - SAMPLE

$$\Rightarrow \int_{A_\varepsilon} k(r^3 - 3t) \vec{u} \cdot \vec{n} dA = -4\pi k(\varepsilon^3 - 3t)$$

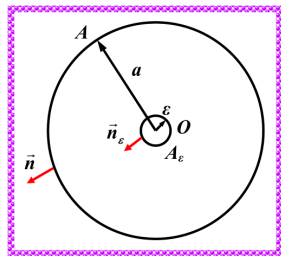
$$\Rightarrow \int_{A+A_\varepsilon} k(r^3 - 3t) \vec{u} \cdot \vec{n} dA = 4\pi k(a^3 - \varepsilon^3)$$

$$\Rightarrow \lim_{\varepsilon \rightarrow 0} \int_{A+A_\varepsilon} k(r^3 - 3t) \vec{u} \cdot \vec{n} dA = 4\pi k a^3$$

则流体质量的随体导数

$$\Rightarrow \frac{DM}{Dt} = -4\pi k a^3 + 4\pi k a^3 = 0$$

$$\begin{aligned} u &= x/r^3 \\ v &= y/r^3 \\ w &= z/r^3 \\ \rho &= k(r^3 - 3t) \end{aligned}$$



OUTLINE

① DESCRIPTIONS OF FLUID FLOWS

Continuous Medium Hypothesis

Eulerian Description and Lagrangian Description

Streamline, Pathline, Streakline, Streamtube

② MATERIAL SUBSTANTIAL DERIVATIVE

③ REYNOLDS TRANSPORT THEOREM

④ HELMHOLTZ VELOCITY DECOMPOSING THEOREM

Expressions of Strain Rate and Rotation Rate Tensor

Physical Meanings of Strain Rate and Rotation Rate Tensors

Description of Fluid Flows with Vorticity

⑤ PATTERNS AND CLASSIFICATION OF FLUID FLOWS

⑥ CONSTITUTIVE EQUATION

Stress Tensor

Constitutive Equation

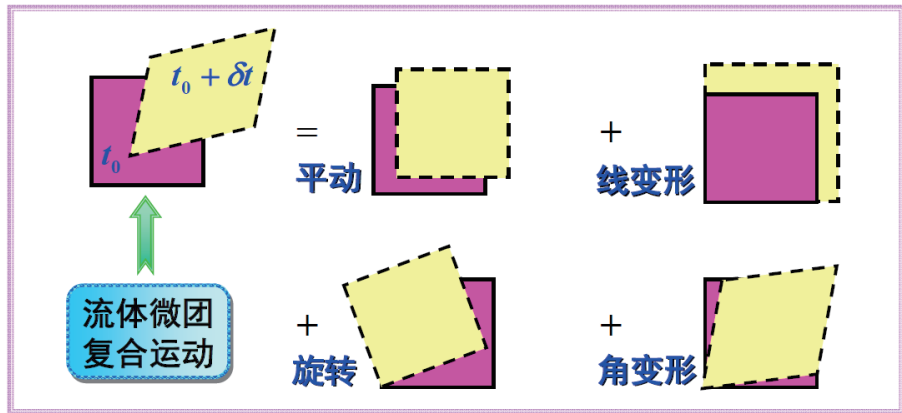
Viscosity Coefficients

HELMHOLTZ DECOMPOSITION

■ Helmholtz速度分解定理

$\Rightarrow$

流体微团的运动可分解为平动、转动、变形三部分之和。



VELOCITY DECOMPOSITION

- $M(x, y, z)$ 处的速度 $\mathbf{u}$
- $M'(x + \delta x, y + \delta y, z + \delta z)$ 处的速度 $\mathbf{u} + \delta \mathbf{u}$

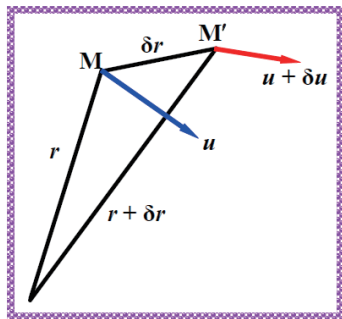
$$\mathbf{u}_{M'} = \mathbf{u}_M + \delta \mathbf{u}$$

- 由Taylor级数展开并略去高阶项

$$\Rightarrow \delta u = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial z} \delta z$$

$$\Rightarrow \delta v = \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y + \frac{\partial v}{\partial z} \delta z$$

$$\Rightarrow \delta w = \frac{\partial w}{\partial x} \delta x + \frac{\partial w}{\partial y} \delta y + \frac{\partial w}{\partial z} \delta z$$



$$\begin{pmatrix} \delta u \\ \delta v \\ \delta w \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

VELOCITY DECOMPOSITION

在 x 方向: $\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$ 分别 $\pm \frac{1}{2} \frac{\partial v}{\partial x}\delta y$, $\pm \frac{1}{2} \frac{\partial w}{\partial x}\delta z$

$$\Rightarrow \delta u = \frac{\partial u}{\partial x}\delta x + \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \delta y + \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \delta z \\ + \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \delta z - \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta y$$

令 (应变率张量的分量)

$$s_{xx} = \frac{\partial u}{\partial x}, \quad s_{xy} = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \quad s_{xz} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right).$$

同时, 令 (旋转角速度)

$$\omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right), \quad \omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right).$$

VELOCITY DECOMPOSITION

$$\Rightarrow \delta u = s_{xx}\delta x + s_{xy}\delta y + s_{xz}\delta z + \omega_y\delta z - \omega_z\delta y$$

同理，对于 y, z 方向的速度分量，有

$$\Rightarrow \delta v = s_{xy}\delta x + s_{yy}\delta y + s_{yz}\delta z + \omega_z\delta x - \omega_x\delta z$$

$$\Rightarrow \delta w = s_{xz}\delta x + s_{yz}\delta y + s_{zz}\delta z + \omega_x\delta y - \omega_y\delta x$$

其中（应变率张量的分量）

$$s_{yy} = \frac{\partial v}{\partial y}, s_{zz} = \frac{\partial w}{\partial z}, s_{yz} = \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right).$$

及（旋转角速度）

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right).$$

VELOCITY DECOMPOSITION

- 流体微团的运动可分解为平动、转动、变形（线变形和角变形）三部分之和。

① 平动速度 $\Rightarrow \mathbf{u}(x, y, z, t)$

② 线变形率 $\Rightarrow \dot{\gamma}_{xx} = \frac{\partial u}{\partial x}, \dot{\gamma}_{yy} = \frac{\partial v}{\partial y}, \dot{\gamma}_{zz} = \frac{\partial w}{\partial z}$

③ 角变形率 $\Downarrow$

$$\dot{\gamma}_{xy} = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \dot{\gamma}_{xz} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right), \dot{\gamma}_{yz} = \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right).$$

④ 旋转角速度 $\Downarrow$

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right), \omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right), \omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right).$$

VELOCITY DECOMPOSITION

- 流体微团的运动可分解为平动、转动、变形（线变形和角变形）三部分之和。写成矩阵形式：

$$\begin{pmatrix} u_{M'} \\ v_{M'} \\ w_{M'} \end{pmatrix} = \begin{pmatrix} u_M \\ v_M \\ w_M \end{pmatrix} + \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} + \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

VELOCITY DECOMPOSITION

- 流体微团的运动可分解为平动、转动、变形（线变形和角变形）三部分之和。写成矩阵形式：

$$\begin{pmatrix} u_{M'} \\ v_{M'} \\ w_{M'} \end{pmatrix} = \begin{pmatrix} u_M \\ v_M \\ w_M \end{pmatrix} + \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} + \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

- 应变率张量

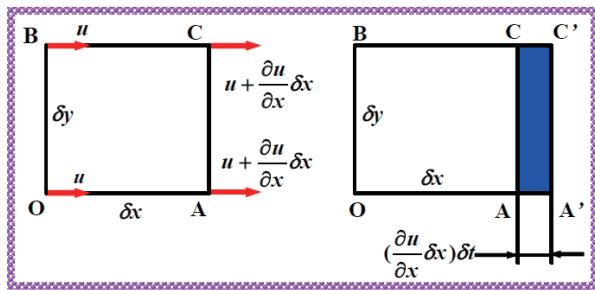
$$S = s_{ij} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}$$

- 旋转率张量

$$A = a_{ij} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

LINEAR DEFORMATION

■ 线变形：体积发生变化



- x 方向的相对变形率 $\Rightarrow \frac{\partial u}{\partial x}$
- y, z 方向的相对变形率 $\Rightarrow \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}$

LINEAR DEFORMATION

- 线变形引起的相对体积总膨胀率

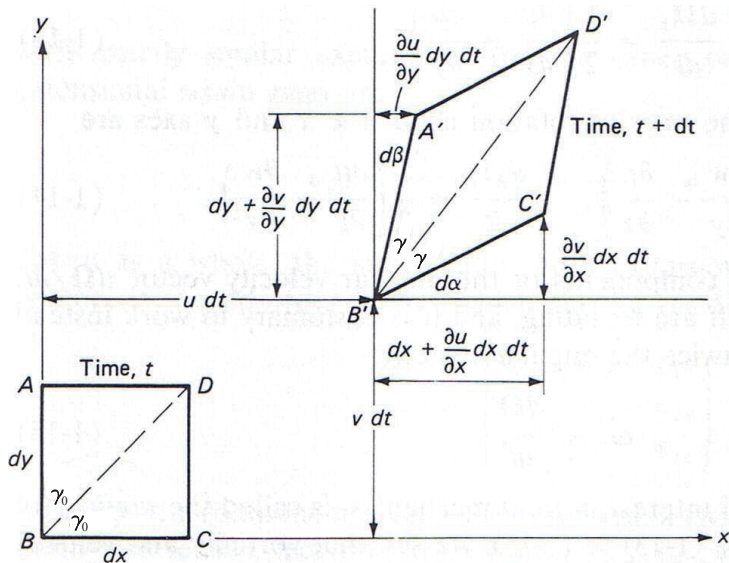


$$\frac{1}{\delta V} \frac{d(\delta V)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{u}$$


单位体积流体单位时间内增加/减少了多少体积，或
单位时间有多少流体体积从单位体积内流入/流出

- 不可压缩流体 $\Rightarrow \nabla \cdot \mathbf{u} = 0$ 速度散度为零

流体微团运动的几何关系



流体微团运动的几何关系

$$d\alpha = \lim_{dt \rightarrow 0} \left(\tan^{-1} \frac{\frac{\partial v}{\partial x} dx dt}{dx + \frac{\partial u}{\partial x} dx dt} \right) = \frac{\partial v}{\partial x} dt$$

$$d\beta = \lim_{dt \rightarrow 0} \left(\tan^{-1} \frac{\frac{\partial u}{\partial y} dy dt}{dy + \frac{\partial v}{\partial y} dy dt} \right) = \frac{\partial u}{\partial y} dt$$

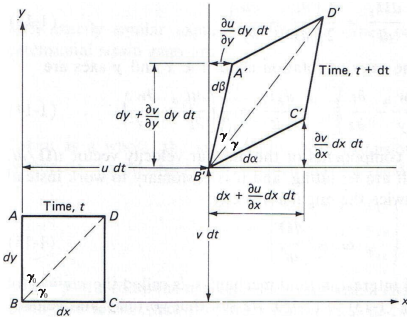
$$\Rightarrow \begin{cases} d\alpha = \frac{\partial v}{\partial x} dt \\ d\beta = \frac{\partial u}{\partial y} dt \end{cases}$$

线段BD的变形角度

$$\delta\gamma = \frac{1}{2} |2(\gamma - \gamma_0)| = \frac{1}{2} (d\alpha + d\beta)$$

角变形, 剪应变率 Shear strain rate

$$S_{xy} = \frac{1}{2} \left(\frac{d\alpha}{dt} + \frac{d\beta}{dt} \right) = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

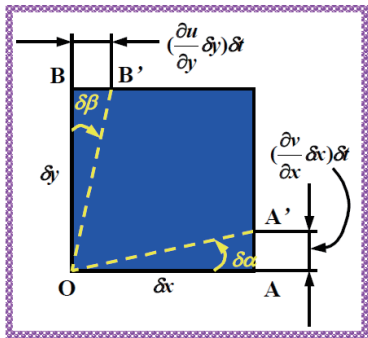


ANGULAR DEFORMATION

② OA、OB (x、y轴) 间的角变形率

$$\dot{\gamma} = \lim_{\delta t \rightarrow 0} \frac{\delta \gamma}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{\delta \alpha + \delta \beta}{2\delta t}.$$

$$\frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = S_{xy} = S_{yx}$$



③ y、z 轴及 z、x 轴间的角变形率

$$S_{xz} = S_{zx} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right), S_{yz} = S_{zy} = \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right).$$

流体微团运动的几何关系

$$d\alpha = \lim_{dt \rightarrow 0} \left(\tan^{-1} \frac{\frac{\partial v}{\partial x} dx dt}{dx + \frac{\partial u}{\partial x} dx dt} \right) = \frac{\partial v}{\partial x} dt$$

$$d\beta = \lim_{dt \rightarrow 0} \left(\tan^{-1} \frac{\frac{\partial u}{\partial y} dy dt}{dy + \frac{\partial v}{\partial y} dy dt} \right) = \frac{\partial u}{\partial y} dt$$

$$\Rightarrow \begin{cases} d\alpha = \frac{\partial v}{\partial x} dt \\ d\beta = \frac{\partial u}{\partial y} dt \end{cases}$$

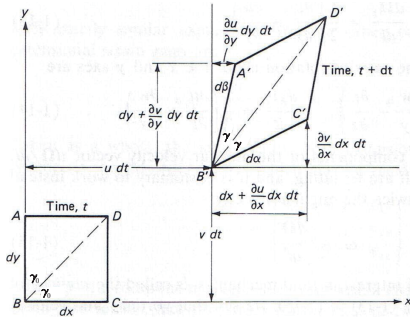
对角线BD的旋转

$$d\Omega_z = \gamma + d\alpha - \gamma_0$$

$$2\gamma + d\alpha + d\beta = 2\gamma_0$$

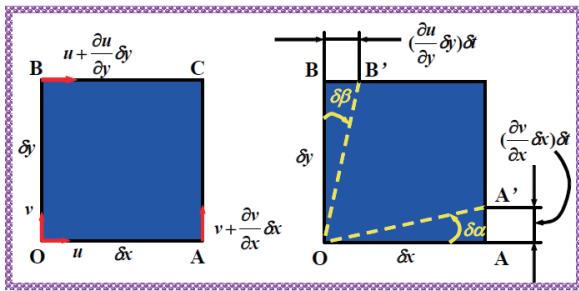
$$\Rightarrow d\Omega_z = \frac{1}{2}(d\alpha - d\beta)$$

$$\Rightarrow \omega_z = \frac{d\Omega_z}{dt} = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$



ROTATION AND VORTICITY

- 旋转：存在交叉导数



④ OA边旋转角速度 $\Rightarrow \omega_{OA} = \lim_{\delta t \rightarrow 0} \frac{\delta \alpha}{\delta t} = \frac{\partial v}{\partial x}$

④ OB边旋转角速度 $\Rightarrow \omega_{OB} = \lim_{\delta t \rightarrow 0} \frac{\delta \beta}{\delta t} = \frac{\partial u}{\partial y}$

ROTATION AND VORTICITY

- 规定：相互垂直的流体线OA和OB的角速度 ω_{OA} 和 ω_{OB} 的平均值为流体微团绕 z 轴的旋转角速度，逆时针方向为正。

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

那么，流体微团绕 x, z 轴的旋转角速度为

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right), \omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)$$

涡量

$\Rightarrow$

$$\vec{\omega} = \frac{1}{2} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{1}{2} \nabla \times \vec{u}$$

ROTATION AND VORTICITY

- 规定：相互垂直的流体线OA和OB的角速度 ω_{OA} 和 ω_{OB} 的平均值为流体微团绕 z 轴的旋转角速度，逆时针方向为正。

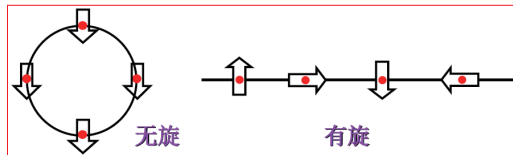
$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

那么，流体微团绕 x, z 轴的旋转角速度为

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right), \omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)$$

无旋流动 $\Rightarrow \omega = 0$

流体微团是否旋转？



有旋流动的判别

例：圆管中的层流流动，当取管轴线与 x 轴重合时，其流速特性为

$$u = u_0 - k(y^2 + z^2), v = 0, w = 0$$

其中 u_0 为管中心速度， k 为常数。试判别该流动是有旋流动还是无旋流动。

解：

$$\omega_x = \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = \frac{1}{2} \times (0 - 0) = 0$$

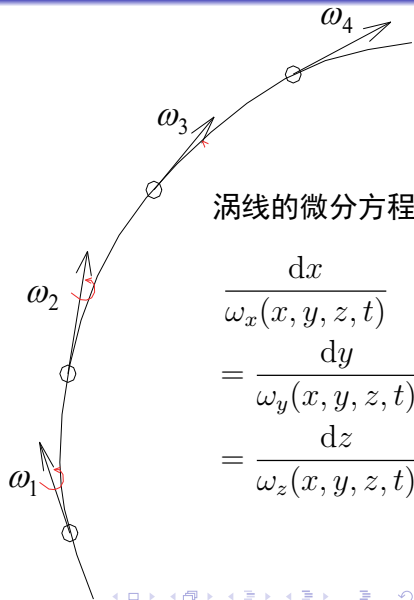
$$\omega_y = \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) = \frac{1}{2}(-2kz - 0) = -kz$$

$$\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{2}(0 + 2ky) = ky$$

所以该圆管中的层流流动是有旋流动。

涡线、涡管、涡束、涡通量

- ① 涡线：一条曲线，在给定瞬时，这条曲线上每一点的切线与位于该点的流体微团的角速度的方向相重合。
- ② 涡管：在给定瞬时，在涡量场中任取一不是涡线的封闭曲线，通过封闭曲线上每一点作涡线，这些涡线形成一个管状表面。
- ③ 涡束：涡管中充满着作旋转运动的流体。



涡线的微分方程

$$\begin{aligned} \frac{dx}{\omega_x(x, y, z, t)} \\ &= \frac{dy}{\omega_y(x, y, z, t)} \\ &= \frac{dz}{\omega_z(x, y, z, t)} \end{aligned}$$

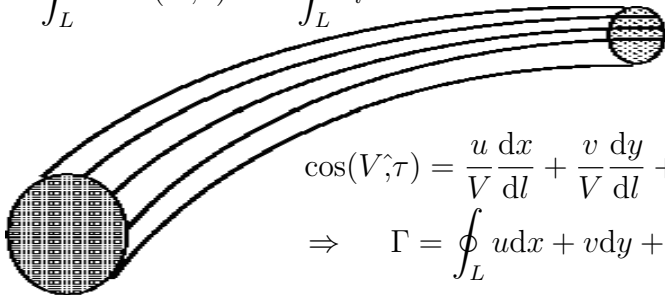
涡通量与速度环量

- ④ 涡通量：旋转角速度的值与垂直于角速度方向的微元涡管横截面积的乘积的两倍。

$$dJ = 2\omega dA, \quad J = 2 \int_{A_0} \omega_n dA$$

- ⑤ 速度环量：流体运动速度绕流体微团的线积分， Γ 。

$$\Gamma \equiv \oint_L V \cos(V, \tau) dl = \oint_L V_l dl$$



$$\begin{aligned} \cos(V, \tau) &= \frac{u}{V} \frac{dx}{dl} + \frac{v}{V} \frac{dy}{dl} + \frac{w}{V} \frac{dz}{dl} \\ \Rightarrow \Gamma &= \oint_L u dx + v dy + w dz \end{aligned}$$

VELOCITY DECOMPOSITION · EXAMPLE

例：平面简单剪切运动的速度分布为： $u = ay$ ， $v = w = 0$
 求：(1) 流体微团的旋转角速度；(2) 应变率张量 s_{ij} ；(3)
 旋转率张量 a_{ij} ；(4) 变形速度 $s_{ij} \delta x_j$ 及旋转速度 $a_{ij} \delta x_j$

(1) 旋转角速度

$$\vec{\omega} = \frac{1}{2} \nabla \times \vec{u} = \frac{1}{2} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = -\frac{1}{2} \underline{\underline{a \vec{k}}}$$

VELOCITY DECOMPOSITION · EXAMPLE

(2) 应变率张量 

$$s_{ij} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \frac{\partial v}{\partial y} & \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & \frac{\partial w}{\partial z} \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & a/2 & 0 \\ a/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}}$$

(3) 变形速度 

$$s_{ij} \delta x_j = \begin{pmatrix} 0 & a/2 & 0 \\ a/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \underline{\underline{\begin{pmatrix} a\delta y/2 \\ a\delta x/2 \\ 0 \end{pmatrix}}}$$

VELOCITY DECOMPOSITION · EXAMPLE

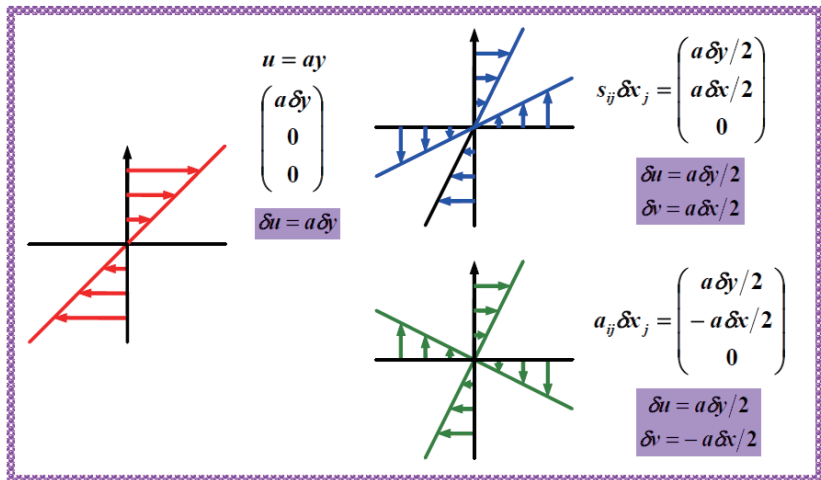
(4) 旋转率张量

$$a_{ij} = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & a/2 & 0 \\ -a/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}}$$

(5) 旋转速度  $a_{ij}\delta x_j = \begin{pmatrix} 0 & a/2 & 0 \\ -a/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \underline{\underline{\begin{pmatrix} a\delta y/2 \\ -a\delta x/2 \\ 0 \end{pmatrix}}}$

VELOCITY DECOMPOSITION · EXAMPLE

⇒ 一个平面剪切运动可分解为 一个剪切变形运动 和一个旋转运动。



OUTLINE

① DESCRIPTIONS OF FLUID FLOWS

Continuous Medium Hypothesis

Eulerian Description and Lagrangian Description

Streamline, Pathline, Streakline, Streamtube

② MATERIAL SUBSTANTIAL DERIVATIVE

③ REYNOLDS TRANSPORT THEOREM

④ HELMHOLTZ VELOCITY DECOMPOSING THEOREM

Expressions of Strain Rate and Rotation Rate Tensor

Physical Meanings of Strain Rate and Rotation Rate Tensors

Description of Fluid Flows with Vorticity

⑤ PATTERNS AND CLASSIFICATION OF FLUID FLOWS

⑥ CONSTITUTIVE EQUATION

Stress Tensor

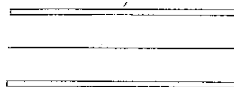
Constitutive Equation

Viscosity Coefficients

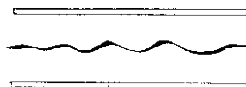
黏性流体的流动形态

雷诺实验：实验现象

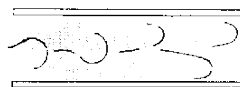
- ① 层流：整个流场呈一簇互相平行的流线。着色流束为一条明晰细小的直线。
- ② 过渡状态：流体质点的运动处于不稳定状态。着色流束开始振荡。



层流



过渡状态

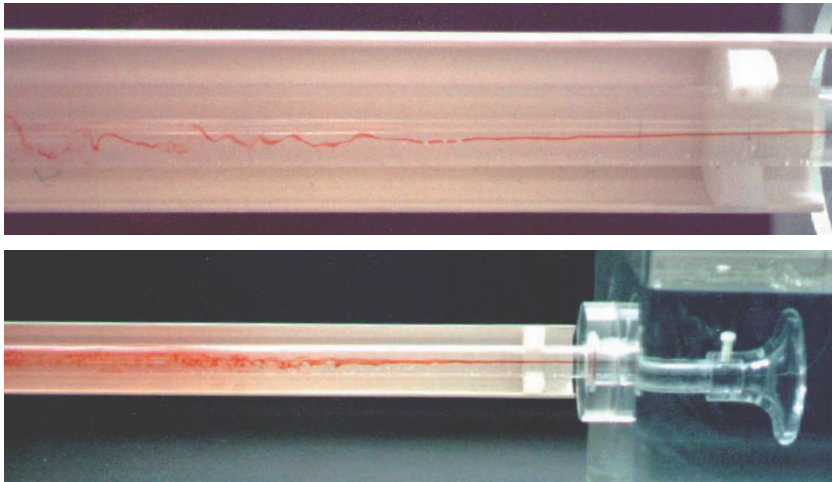


湍流

- ③ 湍流：流体质点作复杂的无规则的运动。着色流束与周围流体相混，颜色扩散至整个玻璃管。

黏性流体的流动形态

雷诺实验：实验现象



黏性流体的流动形态

雷诺准则：两种流动状态的判定

① 实验发现

$v < v_{cr}$ 流动较稳定

$v > v_{cr}$ 流动不稳定

② 临界流速

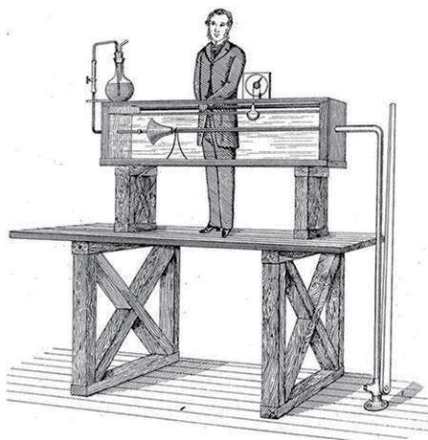
v_{cr} 下临界流速

v'_{cr} 上临界流速

层流： $v < v_{cr}$

不稳定流： $v_{cr} < v < v'_{cr}$

紊流： $v > v'_{cr}$



黏性流体的流动形态

雷诺准则：两种流动状态的判定

③ 临界雷诺数

雷诺数

$$Re = \frac{\rho v d}{\mu} = \frac{v d}{\nu}$$

$$Re_{cr} = 2320 \quad \text{下临界雷诺数}$$

$$Re'_{cr} = 13800 \quad \text{上临界雷诺数}$$

层流：

$$Re < Re_{cr}$$

不稳定流：

$$Re_{cr} < Re < Re'_{cr}$$

紊流：

$$Re > Re'_{cr}$$

工程上常用的圆管临界雷诺数

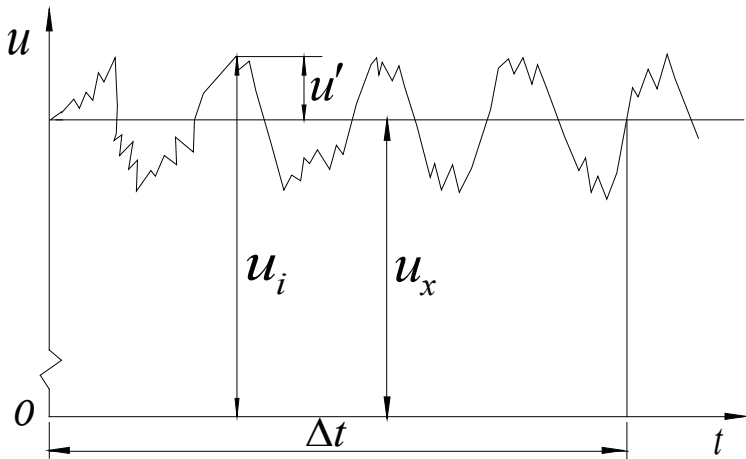
$$Re_{cr} = 2000$$

$$\text{层流：} \quad Re < 2000$$

$$\text{湍流：} \quad Re > 2000$$

黏性流体的流动形态

时均速度 $u(t) = \frac{1}{\Delta t} \int_0^{\Delta t} u_i(t) dt$, 瞬时速度 $u_i = u + u'$



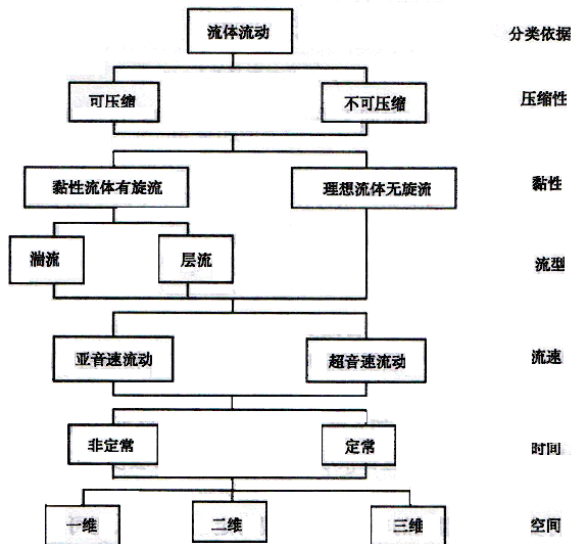
流体流动的分类

- ① 定常流动：流动参量不随时间变化的流动

流场内的速度、压强、密度等参量只是坐标的函数，而与时间无关。

- ② 非定常流动：流动参量随时间变化的流动

流场内的速度、压强、密度等参量不仅是坐标的函数，且与时间有关。



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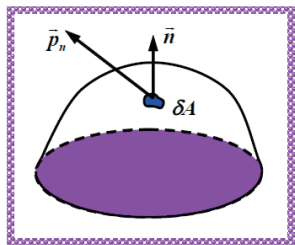
Viscosity Coefficients

STRESS TENSOR · STRESS VECTOR

应力矢量



$$\vec{p}_n = \lim_{\Delta A \rightarrow 0} \frac{\Delta \vec{F}}{\Delta A}$$



下标 n 表示面元 ΔA 的法线方向

② 应力矢量方向与法线方向不一定重合

② 作用在方位各异的面上的应力矢量一般并不相同

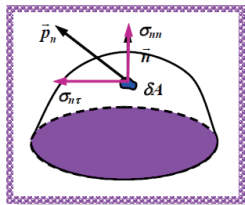
$$\vec{p}_n = \vec{p}_n(\vec{r}, t, \vec{n})$$

STRESS TENSOR · STRESS VECTOR

④ 应力矢量方向与法线方向不一定重合

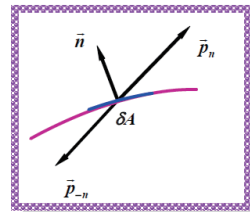
$$\vec{p}_n = (\sigma_{nn}, \sigma_{n\tau})$$

$$\sigma_{nn} = \vec{p}_n \cdot \vec{n} \quad \sigma_{n\tau} = \sqrt{|\vec{p}_n|^2 - \sigma_{nn}^2}$$



④ 应力矢量可以向三个坐标方向分解

$$\vec{p}_n = \sigma_{nx} \vec{i} + \sigma_{ny} \vec{j} + \sigma_{nz} \vec{k}$$



④ 作用在流体两侧的应力矢量大小相等，方向相反

$$\vec{p}_{-n} = -\vec{p}_n$$

STRESS TENSOR · STRESS VECTOR

过空间一点的三个相互垂直平面上的应力矢量
或它们的九个分量完全描写了一点的应力状态

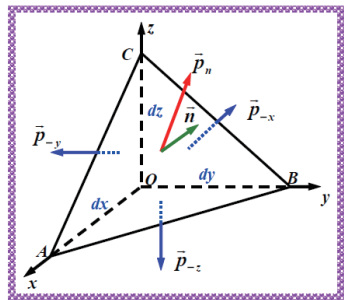
倾斜面外法线单位矢量

$$\begin{aligned}\vec{n} &= \cos(n, x)\vec{i} + \cos(n, y)\vec{j} + \cos(n, z)\vec{k} \\ &= n_x\vec{i} + n_y\vec{j} + n_z\vec{k}\end{aligned}$$

倾斜面和其余三个面的面积

$$\delta A, n_x\delta A, n_y\delta A, n_z\delta A$$

倾斜面和其余三个面上作用的应
力矢量 $\vec{p}_n, \vec{p}_{-x}, \vec{p}_{-y}, \vec{p}_{-z}$



四面体流体微元

STRESS TENSOR · D'ALEMBERT PRINCIPLE

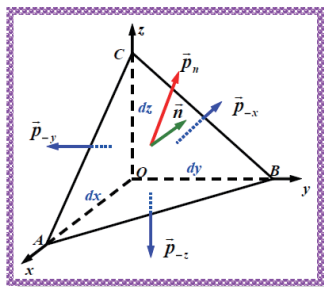
达朗贝尔原理：作用于四面体上的质量力（重力），表面力和惯性力及其力矩应该平衡

惯性力 $\longrightarrow -\rho \vec{a} \delta\tau$ } 三阶无穷小量

重力 $\longrightarrow \rho \vec{g} \delta\tau$

表面力 $\searrow$ 二阶无穷小量

$$(\vec{p}_n + n_x \vec{p}_{-x} + n_y \vec{p}_{-y} + n_z \vec{p}_{-z}) \delta A$$



$\delta\tau \rightarrow 0 \longrightarrow$ 忽略惯性力、重力，只考虑表面力

STRESS TENSOR · D'ALEMBERT PRINCIPLE

$$\Sigma \vec{F} = 0$$

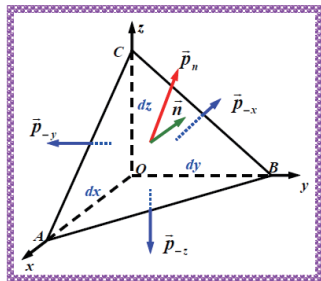
$$\vec{p}_n + n_x \vec{p}_{-x} + n_y \vec{p}_{-y} + n_z \vec{p}_{-z} = 0$$

由

$$\vec{p}_{-x} = -\vec{p}_x, \quad \vec{p}_{-y} = -\vec{p}_y, \quad \vec{p}_{-z} = -\vec{p}_z$$



$$\vec{p}_n = n_x \vec{p}_x + n_y \vec{p}_y + n_z \vec{p}_z$$



达朗贝尔原理，与牛顿第二定律相似，但其发展在于可以把动力学问题转化为静力学问题处理，还可以用平面静力的方法分析刚体的平面运动，这一原理使一些力学问题的分析简单化，而且为分析力学的创立打下了基础。

STRESS TENSOR · D'ALEMBERT PRINCIPLE

- 一点的应力状态可由过空间一点的三个相互垂直平面上的应力矢量描述

由 $\vec{p}_x = \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j} + \sigma_{xz}\vec{k}$

$$\begin{aligned}\vec{p}_n &= n_x \vec{p}_x + n_y \vec{p}_y + n_z \vec{p}_z \\ \vec{p}_n &= \sigma_{nx}\vec{i} + \sigma_{ny}\vec{j} + \sigma_{nz}\vec{k}\end{aligned}$$

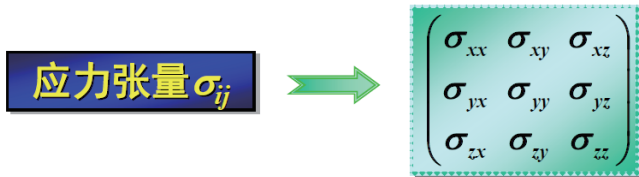
$$\vec{p}_y = \sigma_{yx}\vec{i} + \sigma_{yy}\vec{j} + \sigma_{yz}\vec{k} \quad \vec{p}_z = \sigma_{zx}\vec{i} + \sigma_{zy}\vec{j} + \sigma_{zz}\vec{k}$$

$$\begin{aligned}\vec{p}_n &= \underbrace{(n_x \sigma_{xx} + n_y \sigma_{yx} + n_z \sigma_{zx})}_{\sigma_{nx}} \vec{i} + \underbrace{(n_x \sigma_{xy} + n_y \sigma_{yy} + n_z \sigma_{zy})}_{\sigma_{ny}} \vec{j} \\ &\quad + \underbrace{(n_x \sigma_{xz} + n_y \sigma_{yz} + n_z \sigma_{zz})}_{\sigma_{nz}} \vec{k}\end{aligned}$$

$$\begin{aligned}\vec{p}_n &= (\sigma_{nx} \quad \sigma_{ny} \quad \sigma_{nz}) = (n_x \quad n_y \quad n_z) \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\sigma_{ni} &= n_j \sigma_{ji} \\ \vec{p}_n &= \vec{n} \cdot \Sigma\end{aligned}$$

STRESS TENSOR · ELEMENTS



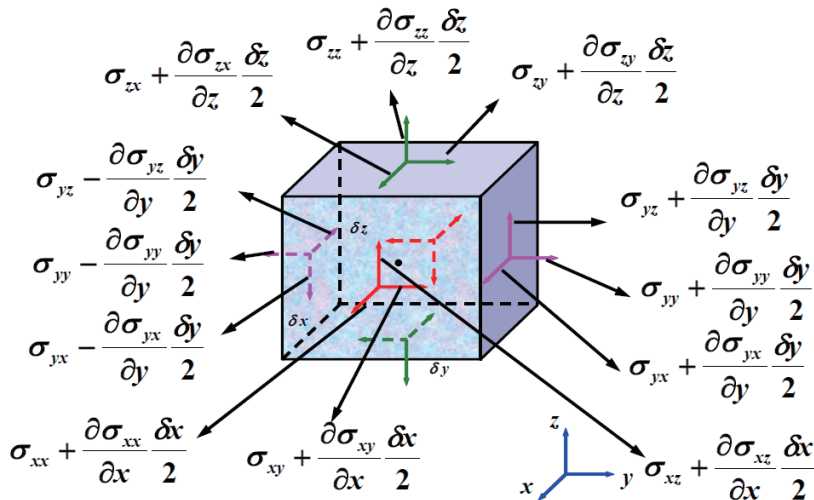
The diagram illustrates the stress tensor. On the left, a blue box contains the text "应力张量 σ_{ij} ". A green arrow points from this box to a green-bordered box on the right. Inside the green box is a 3x3 matrix representing the stress tensor:

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

- ④ 应力张量完全表达了一点应力状态，对角线元素为法向应力分量，非对角线元素为切向应力分量
- ④ 应力张量与作用面方位无关，只是空间点位置和时间的函数 $\sigma_{ni}(\vec{r}, t, \vec{n}) = n_j \sigma_{ji}(\vec{r}, t)$
- ④ 求一点沿某个方位的应力矢量，只需用作用面的单位矢量与该点的应力张量点乘即可

STRESS TENSOR · ELEMENTS

应力张量的 9 个分量中只有 6 个是独立的



STRESS TENSOR · ELEMENTS

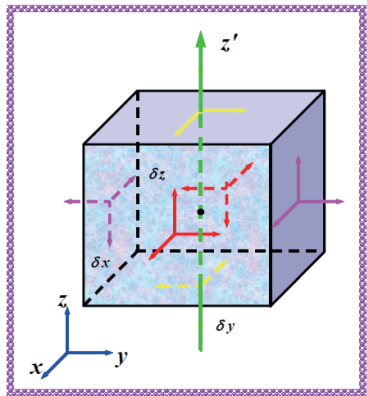
上下表面

表面力对 z' 轴的力矩为0

侧面的四个表面

法向应力作用线的延长线通过 z' 轴，力矩为0

与 z' 轴平行的四个切向应力，力矩为0



STRESS TENSOR · ELEMENTS

只有四个切向应力对 z' 轴
有矩，合力矩为零

$$\left(\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x} \frac{\delta x}{2} \right) \delta y \delta z \frac{\delta x}{2}$$

$$+ \left(\sigma_{xy} - \frac{\partial \sigma_{xy}}{\partial x} \frac{\delta x}{2} \right) \delta y \delta z \frac{\delta x}{2}$$

$$= \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} \frac{\delta y}{2} \right) \delta z \delta x \frac{\delta y}{2} + \left(\sigma_{yx} - \frac{\partial \sigma_{yx}}{\partial y} \frac{\delta y}{2} \right) \delta z \delta x \frac{\delta y}{2}$$



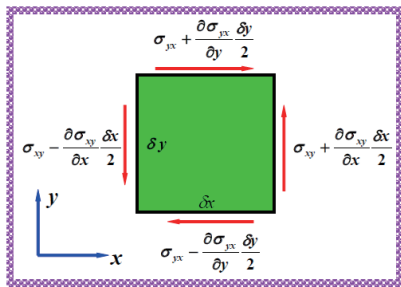
$$\sigma_{xy} = \sigma_{yx}$$

同理

$$\sigma_{xz} = \sigma_{zx}$$

$$\sigma_{yz} = \sigma_{zy}$$

应力张量是
二阶对称张量



STRESS TENSOR · ELEMENTS

理想流体与静止流体的应力张量

- 理想流体或静止流体中切应力为零，只有法向应力且同一点各个不同方向上的法向应力相等

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{nn} = -p$$

$$\sigma_{ij} = -p\delta_{ij}$$

- 负号是强调压强与作用面的法线方向相反
- 在理想流体或静止流体中，只用一个标量函数即压强便完全地描述了一点上的应力状态

CONSTITUTIVE EQUATION

■ 牛顿流体的本构方程 Constitutive Equation

描述牛顿流体应力与应变率之间关系的方程。

■ 斯托克斯假设 Stokes' Hypothesis

- ① “线性函数” 假设：
应力张量是应变率张量的线性函数。
- ② “各向同性” 假设：
流体各向同性，其性质与方向无关。
- ③ “零应变率” 假设：
流体静止时的应变率为零，其应力就是静压强。

CONSTITUTIVE EQUATION

- 根据“线性函数”假设，应力张量可写为

$$\Sigma = aS + bI$$

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = a \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \frac{\partial v}{\partial y} & \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & \frac{\partial w}{\partial z} \end{pmatrix} + b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

由牛顿内摩擦定理, **Newton's Law of Viscosity**: $\tau = \mu \frac{du}{dy}$ (1-side)

$$\Rightarrow \sigma_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \mu \cdot 2s_{12}, \text{ and } \sigma_{xy} = a s_{12} \Rightarrow a = 2\mu.$$

CONSTITUTIVE EQUATION

由右式可得

$$\sigma_{xx} = 2\mu \frac{\partial u}{\partial x} + b$$

$$\sigma_{yy} = 2\mu \frac{\partial v}{\partial y} + b$$

$$\sigma_{zz} = 2\mu \frac{\partial w}{\partial z} + b$$

$$\Rightarrow \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 2\mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 3b = 2\mu \nabla \cdot \vec{u} + 3b$$

$$\Rightarrow b = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) - \frac{2}{3}\mu \nabla \cdot \vec{u}$$

$$\Rightarrow \Sigma = 2\mu S + \left[\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) - \frac{2}{3}\mu \nabla \cdot \vec{u} \right] I$$

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = a \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) & \frac{\partial v}{\partial y} & \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) & \frac{1}{2} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & \frac{\partial w}{\partial z} \end{pmatrix} + b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

CONSTITUTIVE EQUATION

- 牛顿流体应力张量与应变率张量之间的关系（Stokes假定）
 - 偏应力张量 τ_{ij} 各分量是局部速度梯度张量 $\partial u_i / \partial x_j$ 各分量的线性齐次函数。

$$\tau_{ij} = \alpha_{ijkl} \frac{\partial u_k}{\partial x_l}$$

- 流体的物理性质与方向无关且不依赖于坐标系的选择和变换。

$$\alpha_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{il} \delta_{jk} + \mu' \delta_{il} \delta_{jk}$$

- 运动流体的应力张量在运动停止后应趋于静止流体的应力张量。

$$\sigma_{ij} = -p \delta_{ij} + \tau_{ij} = -p \delta_{ij} + \lambda \delta_{ij} s_{kk} + 2\mu s_{ij}$$

- ▶ 根据假设2、3（“各向同性”假设、“零应变率”假设）可得

$$\Sigma = -p \mathbf{I} \quad \text{for static fluids}$$



CONSTITUTIVE EQUATION

- 定义一点的平均法向应力如下

$$\bar{p} = -\frac{1}{3}\sigma_{ii} = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}), \quad -\bar{p} = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}),$$

一般情况下, $\bar{p} \neq p$ 。两者之差为与由流动引起的应力项相关:

$$\begin{aligned} -\bar{p} &= \frac{1}{3}[(-p + \lambda s_{kk} + 2\mu s_{xx}) + (-p + \lambda s_{kk} + 2\mu s_{yy}) + (-p + \lambda s_{kk} + 2\mu s_{zz})] \\ &= -p + \lambda s_{kk} + \frac{2}{3}\mu s_{kk} = -p + \left(\lambda + \frac{2}{3}\mu\right) s_{kk} \quad \Rightarrow \quad p - \bar{p} = K s_{kk} \end{aligned}$$

即平均法向应力与 p 之差正比于速度矢量的散度 s_{kk} , 也可表示为

$$\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = -p + K \nabla \cdot \vec{u} \quad (2)$$

其比例系数成为**体积黏性系数**

$$K = \lambda + \frac{2}{3}\mu$$

CONSTITUTIVE EQUATION

把

$$\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = -p + K\nabla \cdot \vec{u}$$

代入

$$\Sigma = 2\mu\mathbf{S} + \left[\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) - \frac{2}{3}\mu\nabla \cdot \vec{u} \right] \mathbf{I}$$

可得

$$\Sigma = [-p + \lambda\nabla \cdot \vec{u}] \mathbf{I} + 2\mu\mathbf{S}, \quad \lambda = K - \frac{2}{3}\mu \quad (3)$$

也可写成

$$\sigma_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (4)$$

CONSTITUTIVE EQUATION

写成分量形式

广义牛顿应力公式



$$\begin{aligned}\sigma_{xx} &= -p + 2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \vec{u} \\ \sigma_{yy} &= -p + 2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot \vec{u} \\ \sigma_{zz} &= -p + 2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot \vec{u}\end{aligned}$$

$$\begin{aligned}\sigma_{xy} &= \sigma_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \\ \sigma_{xz} &= \sigma_{zx} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \\ \sigma_{yz} &= \sigma_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)\end{aligned}$$

- ☞ 压强，各向同性
- ☞ 由体积膨胀或收缩引起的各向同性粘性应力
- ☞ 由运动流体微团变形引起的粘性应力

VISCOSITY COEFFICIENT

体积粘性系数 K 和第二粘性系数 λ

定义平均法向应力

$$\bar{p} = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

$$\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = -p + K\nabla \cdot \vec{u}$$

$$\lambda = K - \frac{2}{3}\mu$$



$$\bar{p} = p - K\nabla \cdot \vec{u} = p - \left(\lambda + \frac{2}{3}\mu\right)\nabla \cdot \vec{u}$$

不可压缩流体

$$\nabla \cdot \vec{u} = 0$$

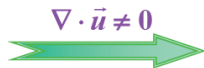


$$\bar{p} = p$$

☞ 平均法向应力等于平衡态压强，应力张量中 λ 自动不出现

VISCOSITY COEFFICIENT

可压缩流体



$$\bar{p} = p - \underbrace{\left(\lambda + \frac{2}{3} \mu \right)}_K \nabla \cdot \vec{u}$$

☞ 流体发生膨胀或压缩时，平均法向应力不等于平衡态压强，且 λ 出现在与相对体积膨胀率有关的项中

② 体积膨胀引起粘性应力与体积变化时的能量耗散机制有关，除高温和高频声波等极端情况，一般气体运动可近似认为

$$K = \lambda + \frac{2}{3} \mu = 0 \quad \text{即} \quad \lambda = -\frac{2}{3} \mu$$



斯托克斯假设

VISCOSITY COEFFICIENT



$$\Sigma = 2\mu S + \left[-p - \frac{2}{3}\mu \nabla \cdot \vec{u} \right] I$$

$$\sigma_{ij} = -p\delta_{ij} - \frac{2}{3}\delta_{ij}\mu \frac{\partial u_k}{\partial x_k} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\begin{aligned}\sigma_{xx} &= -p + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu \nabla \cdot \vec{u} \\ \sigma_{yy} &= -p + 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu \nabla \cdot \vec{u} \\ \sigma_{zz} &= -p + 2\mu \frac{\partial w}{\partial z} - \frac{2}{3}\mu \nabla \cdot \vec{u} \\ &\vdots\end{aligned}$$



斯托克斯假设下，本构方程中只出现一个粘性系数 μ

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Thanks!
Welcome Questions